Sheaves of modules

Monday, November 4, 2024 5:10 PM

12.11.25 Lecture continued

Def: A category $\mathcal{C}$ is abelian if the following holds

1) For $A, B \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is an abelian group.
2) For $A, B, C \in \mathcal{C}$, $f, g \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, $h \in \mathrm{Hom}_{\mathcal{C}}(B,C)$
   $h\circ(f+g) = h\circ f + h\circ g$
3) There is a zero object $\underline{0}$ [i.e $id_{\underline{0}} = 0 \in \mathrm{Hom}_{\mathcal{C}}(\underline{0}, \underline{0})$]
4) One can take co-product (direct sum) of two objects
5) Every morphism has kernel and cokernel
6) For $f: A \to B$ in $\mathcal{C}$
   $\mathrm{coker}(\ker f \to A) \xrightarrow{\sim} \ker(B \to \mathrm{coker} f)$

Eg: R ring, the category of R-mods is abelian.

Eg: R ring, the category of free R-mods does not satisfy (5).

$(X, \mathcal{O}_X)$ be a ringed space.

Notation. $\mathrm{Mod}_{\mathcal{O}_X}$ = category of sheaves of $\mathcal{O}_X$-mods on X. =: category of $\mathcal{O}_X$-mods.

objects of $\mathrm{Mod}_{\mathcal{O}_X}$ = sheaves of $\mathcal{O}_X$-mods on X

A morphism $f: \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-mods is a map of sheaves s.t $f_{|U}: \mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$ lin for $U \underset{\text{open}}{\subseteq} X$. Such a morphism is called $\underline{\mathcal{O}_X\text{- linear}}$.

e.g: $g: Y \to X$ morphism of schemes
$g_*\mathcal{O}_Y \in \mathrm{Mod}_{\mathcal{O}_X}$, $g^\#: \mathcal{O}_X \to g_*\mathcal{O}_Y$ is $\mathcal{O}_X$-lin by definition.

§ Operation on $\mathrm{Mod}_{\mathcal{O}_X}$.

- Given $\mathcal{O}_X$-linear $f: \mathcal{F} \to \mathcal{G}$, $\ker f$, $\mathrm{coker}\, f$, $\mathrm{im}(f)$ are in $\mathrm{Mod}_{\mathcal{O}_X}$.

- $\mathrm{Mod}_{\mathcal{O}_X}$ admits $\otimes_{\mathcal{O}_X}$ pdt.
  $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) :=$ sheafification of the presheaf
  $U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$

- Admits an internal hom sheaf
  $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$

- Admits an <u>internal hom sheaf</u>

$\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$

- $\mathrm{Mod}_{\mathcal{O}_X}$ admits arbitrary direct sum and product

<u>Ex.</u> Check $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}, \mathcal{H}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{G},\mathcal{H}))$ in $\mathrm{Mod}_{\mathcal{O}_X}$.

<u>Def.</u> $(f,f^{\#}) : (Y,\mathcal{O}_Y) \longrightarrow (X,\mathcal{O}_X)$ map of ringed spaces. $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$. The $f^*\mathcal{G}$ = pull back of $\mathcal{G}$ via f is $f^{-1}\mathcal{G} \otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_Y$

[$\mathcal{O}_X \to f_*\mathcal{O}_Y$, gives $f^{-1}(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y$, The $\otimes$ is taken on the ringed space $(Y, f^{-1}\mathcal{O}_X)$

<u>Rmk.</u> 1) $f^*\mathcal{G}$ ~~is~~ $\in \mathrm{Mod}_{\mathcal{O}_Y}$ (Ex).

2) $(Y,\mathcal{O}_Y)$, $(X,\mathcal{O}_X)$ locally ringed spaces. (eg schemes).

$$(f^*\mathcal{G})_y = \mathcal{G}_{f(y)} \otimes_{\mathcal{O}_{X,f(y)}} \mathcal{O}_{Y,y}$$

<u>Recall.</u> $f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_Y$ induces $\mathcal{O}_{X,f(y)} \longrightarrow \mathcal{O}_{Y,y}$.

§ <u>$\tilde{A}$ construction.</u>

Already covered by Domenico.

Let A be a ring, $M \in \mathrm{Mod}_A$. Consider the presheaf $\tilde{M}$ on $\mathrm{Spec}(A)$.

$$\tilde{M}(U) = \left\{ \text{set maps } s : U \to \coprod_{p\in U} M_p \;\middle|\; \begin{array}{l} (i)\ s(p) \in M_p \\ (ii)\ \text{for } p \in U, \exists\, p \in V \underset{\text{open}}{\subseteq} U \\ \text{and } m \in M \text{ s.t } \\ s(q) = m/f \in M_q \\ \text{for some } f \notin \bigcup_{q\in V} q \end{array} \right\}$$

<u>Thm</u> (Prop 5.1, Hart)

a) $\tilde{M}$ is an $\mathcal{O}_X$-mod.

b) for $p \in \mathrm{Spec}(A)$, $(\tilde{M})_p \xrightarrow{\sim} M_p$ as A_p modules

c) For $f \in A$, The natural map $M_f \xrightarrow{\sim} \tilde{M}(D(f))$ is an isom.

d) In particular $\Gamma(\mathrm{Spec}(A), \tilde{M}) \longleftarrow M$ is an isom.

Pf. (a), (b) clear, note (c) $\Rightarrow$ (d)

(c) Lemma: Let $s \in \Gamma(\tilde{M}, \mathrm{Spec}(A))$ such that $s|_{D(g)} = 0$ for some $g \in A$. Then $g^m \cdot s = 0$ for some $m \in \mathbb{N}_{\geq 0}$.

Pf. Choose $g_1, g_2, \dots g_r$ such that $s = m_i/g_i$ on $D(g_i)$ and $\bigcup_{i=1}^{r} D(g_i) = \mathrm{Spec}(A)$.

Since $s|_{D(g_i) \cap D(g)} = s|_{D(g_i g)} = 0$

$m_i/g_i = 0 \in M_g \quad g \in D(g_i g)$

$\Rightarrow m_i/g_i = 0 \in M_{g_i g} = M_{g_i}[1/g]$

$\Rightarrow g^{t_i} \cdot \frac{m_i}{g_i} = 0$ in A_{g_i} for some t_i

$\Rightarrow g^{t_i} \cdot s|_{D(g_i)} = 0$

Take $m = \max\{t_1, t_2 \dots t_r\}$.

Then $g^m \cdot s|_{D(g_i)} = 0 \ \forall i \Rightarrow g^m \cdot s = 0$ $\square$

Thm, c Given $s \in \tilde{M}(D(f))$, choose $g_1, g_2, \dots, g_r$ such that
- $D(f) = \bigcup_{i=1}^{r} D(g_i)$ and
- On $D(g_i)$, $s = m_i/g_i$ for some $m_i \in M$.

i.e. $g_i s = \frac{m_i}{1}$ on $D(g_i)$

By Lemma $g_i^{n_i+1} s = g_i^{n_i} \cdot m_i \in \Gamma(\mathrm{Spec}\, A[1/f], \tilde{M})$ for some n_i $(*)$

$\bigcup_{i=1}^{r} D(g_i) = \bigcup_{i=1}^{r} D(g_i^{n_i+1}) = D(f)$

$\Rightarrow V(f) = \underline{V(g_1^{n_1+1}, g_2^{n_2+1} \dots, g_r^{n_r+1})}$

$\Rightarrow f \in \sqrt{(g_1^{n_1+1}, \dots g_r^{n_r+1})}$

$\Rightarrow f^m = g_1^{n_1+1}\beta_1 + \dots + g_r^{n_r+1}\beta_r$ for some m & $\beta_1 \dots \beta_r \in A$.

in $\Gamma(D(f), \tilde{M})$

$(*)$ $\Rightarrow f^m s = (\sum g_i^{n_i+1}\beta_i) s \overset{\downarrow}{=} \sum g_i^{n_i}\beta_i \cdot m_i \in M$

$\Rightarrow s \in \mathrm{Im}(M_f \to \tilde{M}(D(f)))$.

Rmk. Given $M \in \mathrm{Mod}_A$, consider the presheaf $\mathcal{M}$ of $\mathcal{O}_X$ modules where $X = \mathrm{Spec}\, A$:

$\mathcal{M}(V) = \{ m/f \mid f \notin \cup\, q,\ q \in V \}$.

Then $\tilde{M}$ is the sheafification of $\mathcal{M}$.

Thm. Given an A-mod map $f: M \to N$,
[illegible] on $\mathcal{O}_{\mathrm{Spec}\, A}$ linear map

Thm. Given an A-mod map $f: M \to N$, one naturally gets an $\mathcal{O}_{\operatorname{Spec} A}$ linear map $\tilde{f}: \tilde{M} \to \tilde{N}$. Moreover $\widetilde{id_M} = id_{\tilde{M}}$ and $\widetilde{f \circ g} = \tilde{f} \circ \tilde{g}$.

So $M \mapsto \tilde{M}$ is a functor from the category of A-modules to the category of $\mathcal{O}_{\operatorname{Spec}(A)}$-modules.

Thm. The functor $M \mapsto \tilde{M}$ above has the following properties.

(i) The natural map $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec}(A)}}(\tilde{M}, \tilde{N})$ is an isom.

(ii) $X = \operatorname{Spec} A$, $\mathcal{F} \in \operatorname{Mod}_{\mathcal{O}_X}$, $\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}) \to \operatorname{Hom}_A(M, \Gamma(X, \mathcal{F}))$ is an isom. Conversely for $\mathcal{G} \in \operatorname{Mod}_{\mathcal{O}_X}$, if the natural map $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}) \xrightarrow{\Psi^{\mathcal{G}}_{\mathcal{F}}} \operatorname{Hom}_A(\Gamma(X, \mathcal{G}), \Gamma(X, \mathcal{F}))$ is an isomorphism, then $\mathcal{G} \xrightarrow{\sim} \tilde{M}$ for some $M \in \operatorname{Mod}_A$.

(iii) $\widetilde{\left(\bigoplus_{i \in I} M_i\right)} = \bigoplus_{i \in I} \tilde{M}_i$ in $\operatorname{Mod}_{\mathcal{O}_{\operatorname{Spec}(A)}}$.

(iv) $\widetilde{M \otimes_A N} \xrightarrow{\sim} \tilde{M} \otimes_{\mathcal{O}_{\operatorname{Spec}(A)}} \tilde{N}$.

(v) $\varphi: A \to B$ ring homo, $M \in \operatorname{Mod}_A$, $(\varphi^\#)^*(\tilde{M}) \xrightarrow{\sim} \widetilde{M \otimes_A B}$ in $\operatorname{Mod}_{\operatorname{Spec} B}$

(vi) φ as in (v), $N \in \operatorname{Mod}_B$, $(\varphi^\#)_* \tilde{N} = \tilde{N}$

where on the right N is considered as an A-mod via restriction of scalars.

Pf (ii) sufficient cond^n ensuring $\mathcal{G} \xrightarrow{\sim} \tilde{M}$.

Take $M = \Gamma(X, \mathcal{G})$. Our assumption $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \tilde{M}) \xrightarrow[\sim]{\Psi^{\mathcal{G}}_{\tilde{M}}} \operatorname{Hom}_A(\Gamma(X, \mathcal{G}), M)$ gives a map $\varphi_1: \mathcal{G} \to \tilde{M}$ such that $\Psi^{\mathcal{G}}_{\tilde{M}}(\varphi_1) = id_M$.

The isom $\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{G}) \xrightarrow[\sim]{\Psi^{\tilde{M}}_{\mathcal{G}}} \operatorname{Hom}_A(M, M)$ gives $\varphi_2: \tilde{M} \to \mathcal{G}$ such that $\Psi^{\tilde{M}}_{\mathcal{G}}(\varphi_2) = id_M$

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{G}) \arrow[r, "\Psi^{\tilde{M}}_{\mathcal{G}}"] \arrow[d, "\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \varphi_1)"'] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{Hom}_A(M, M) \arrow[d, "\operatorname{Hom}(M, id) = id"] \\
\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{M}) \arrow[r, "\Psi^{\tilde{M}}_{\tilde{M}}"'] & \operatorname{Hom}(M, M)
\end{tikzcd}$$

So $\Psi^{\tilde{M}}_{\mathcal{G}}(\varphi_2) = \Psi^{\tilde{M}}_{\tilde{M}}(\varphi_1 \cdot \varphi_2) = id$

Since $\Psi^{\tilde{M}}_{\tilde{M}}$ is an isom $\varphi_1 \cdot \varphi_2 = id$.

Thm. $M, N \in \operatorname{Mod}_A$. $f: M \to N$ A-linear.

Then $\operatorname{Ker} \tilde{f} \cong \widetilde{\operatorname{Ker} f}$, $\widetilde{\operatorname{coker} f} \cong \operatorname{coker} \tilde{f}$

Thm: $M, N \in \mathrm{Mod}_A$. $f: M \to N$ A-linear.
Then $\ker \tilde{f} \simeq \widetilde{\ker f}$, $\widetilde{\mathrm{coker} f} \xleftarrow{\sim} \mathrm{coker}\, \tilde{f}$

Pf. Have $\ker f \to M \to N$. This gives a complex $\widetilde{\ker f} \to \tilde{M} \to \tilde{N}$. This gives a map $\widetilde{\ker f} \to \ker \tilde{f}$, which is an isom at every stalk.

Thm. $0 \to M' \to M \to M'' \to 0$ is exact in Mod_A
$\iff$ $0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ is exact in $\mathrm{Mod}_{\mathcal{O}_X}$.

§ Quasi coherent (QCoh) sheaves (ref stacks project tag 01BD)

$(X, \mathcal{O}_X)$ ringed space, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$

Def. $\mathcal{F}$ is QCoh if for every $x \in X$, $\exists$ a nbhd U of x and an exact seq

$$\bigoplus_{i \in I} \mathcal{O}_U \to \bigoplus_{j \in J} \mathcal{O}_U \to \mathcal{F}|_U \to 0$$

Thm. $X = \mathrm{Spec}(A)$, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$. $\mathcal{G}$ is QCoh iff $\mathcal{G} \simeq \tilde{M}$ in $\mathrm{Mod}_{\mathcal{O}_X}$.

Pf. $\Leftarrow$ Choose a presentation $\bigoplus_{j \in J} A \to \bigoplus_{i \in I} A \to M \to 0$

This gives an exact seq $\bigoplus_{j \in J} \mathcal{O}_X \to \bigoplus_{i \in I} \mathcal{O}_X \to \tilde{M} \to 0$.

$\Rightarrow$ Choose $f_1, f_2, \ldots f_n \in A$ such that $X = \bigcup_{j=1}^{n} D(f_j)$ and for each j, have an exact seq

$$\bigoplus_J \mathcal{O}_{D(f_j)} \to \bigoplus_I \mathcal{O}_{D(f_j)} \to \mathcal{G}|_{D(f_j)} \to 0$$

Since $\mathcal{O}_{D(f_j)} = \widetilde{A[1/f_j]}$, They are QCoh.
So the coker $\mathcal{G}|_{D(f_j)} \xrightarrow{\sim} \widetilde{\Gamma(D(f_j), \mathcal{G})}$
Let $i_j : D(f_j) \hookrightarrow X$, $i_{j_1 j_2} : \underset{\substack{\| \\ D(f_{j_1} f_{j_2})}}{D(f_{j_1}) \cap D(f_{j_2})} \hookrightarrow X$

be the natural open immersions

We have an exact seq

$$0 \to \mathcal{G} \to \bigoplus_j (i_j)_* (\mathcal{G}|_{D(f_j)}) \to \bigoplus (i_{j_1 j_2})_* (\mathcal{G}|_{D(f_{j_1} f_{j_2})})$$

$\bigoplus_j (i_j)_* (\mathcal{G}|_{D(f_j)}) \cong \widetilde{\Gamma(D(f_j), \mathcal{G})}$ on X, $\quad \bigoplus (i_{j_1 j_2})_* (\mathcal{G}|_{D(f_{j_1} f_{j_2})}) \cong \widetilde{\Gamma(D(f_{j_1} f_{j_2}), \mathcal{G})}$ on X

$(s_j) \longmapsto ((s_{j_1} - s_{j_2}))$

$\mathcal{G}$ being the kernel, is also QCoh.

Thm: Let X be a scheme, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$.

Thm: Let X be a scheme, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$.

(i) $\mathcal{G}$ is QCoh iff for every affine open $U \subseteq X$, $\mathcal{G}_{|U} \xrightarrow{\sim} \tilde{M}_U$ for some $M_U \in \mathrm{Mod}_{\mathcal{O}_X(U)}$.

(ii) $\mathcal{G}$ is QCoh iff $\exists$ an affine open covering $X = \cup U_\lambda$ s.t $\mathcal{G}_{|U_\lambda} \xrightarrow{\sim} \tilde{M}_\lambda$ for some $M_\lambda \in \mathrm{Mod}_{\mathcal{O}_X(U_\lambda)}$

(iii) When X is affine checking $\mathcal{G} \xrightarrow{\sim} \tilde{M}$ iff $\mathcal{G}_{|U_\lambda} \xrightarrow{\sim} \tilde{M}_\lambda$ for some affine covr $X = \cup U_\lambda$ and $M_\lambda \in \mathrm{Mod}\, \mathcal{O}_X(U_\lambda)$.

Thm. X be a scheme, The category of QCoh sheaves $\mathrm{QCoh}(X)$ is abelian, admits arbitrary direct sum.

Pf HW:

§ Coherent Sheaves: (abv. coh)

The notion of coh sheaves on a scheme X is defined under The additional assumption X is locally noetherian (be careful, Hart assumes additionally X is noeth, which we do not).

Def. Let X be a locally noetherian scheme. $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$. $\mathcal{G}$ is called coherent (ab coh) if one of The following equivalent condn are satisfied (eq. is HW)

(i) $\mathcal{G}$ is QCoh and for every affine open $U \subseteq X$, $\Gamma(U, \mathcal{G})$ is a finitely generated $\mathcal{O}_X(U)$ mod.

(ii) There is an affine open covering $X = \underset{\lambda}{\cup} U_\lambda$ s.t $\mathcal{G}_{|U_\lambda}$ is QCoh and $\Gamma(U_\lambda, \mathcal{G})$ is a f.g $\mathcal{O}_X(U_\lambda)$ mod for $\forall \lambda$.

(iii) For every affine open $U \subseteq X$ or on each element U_λ of an affine open covering $X = \cup U_\lambda$, $\mathcal{G}_{|U}$ or $\mathcal{G}_{|U_\lambda}$ is isom to $\tilde{M}$ for some f.g $\mathcal{O}_X(U)$ or $\mathcal{O}_X(U_\lambda)$ –mod M.

Thm. $\mathrm{QCoh}(X)$ is closed under taking Ker, coker, $\oplus$, $\otimes$, $\mathrm{QCoh}(X)$ is an abelian category.

Notation $Coh(X)$ = set of all coh $\mathcal{O}_X$-module on X.

Thm. X, Y be noeth schemes.

(i) $\varphi : \mathcal{F} \to \mathcal{G}$ $\mathcal{O}_X$-lin map between coh sheaves
$\ker \varphi$, coker φ coh.

(ii) $f : Y \to X$, $\mathcal{F} \in Coh(X)$, $f^* \mathcal{F} \in Coh(Y)$.

(iii) If f is finite $\mathcal{G} \in Coh(Y)$, $f_* \mathcal{G} \in Coh(X)$.

(iv) $\psi : \mathcal{F} \to \mathcal{G}$ $\mathcal{O}_X$-lin $\mathcal{G} \in QCoh(X)$, $\mathcal{F} \in Coh(X)$.
$\mathrm{Im}\, \psi \in Coh(X)$.

(v) $\mathcal{F}, \mathcal{G} \in Coh(X)$, so is $\mathcal{F} \otimes \mathcal{G}$, $\mathcal{F} \oplus \mathcal{G}$.

End of what is already covered by Domenico.

Def. A morphism of schemes $f : X \to Y$ is called quasi-compact if inverse image of any quasi-compact open subset under f is quasi-compact.

Prop. $f : X \to Y$ is quasi compact $\Leftrightarrow$ $\exists$ an open covering $Y = \bigcup_i V_i$ s.t each V_i is quasi-compact and $f^{-1}(V_i)$ is quasi-compact.

Def: $f : X \to Y$ is called quasi-separated if the diagonal $\Delta : X \to X \times_Y X$ is quasi-compact. A scheme is called quasi-separated if $f : X \to \mathrm{Spec}\, \mathbb{Z}$ is quasi-separated.

Prop S affine. $f : X \to S$ q. separated. $\Leftrightarrow$ for any two affine opens $U, V \subseteq X$, $U \cap V$ is quasi-compact.

Pf. $U \cap V = \Delta^{-1}(U \times_S V)$ and $U \times_S V$ is affine, hence quasi-compact.

Eg: • X noetherian scheme. Every morphism $f : X \to Y$ is quasi-compact
So noetherian schemes are quasi-separated over any base scheme

• Separated morphisms are quasi separated.

Thm. $f : X \to Y$ map of schemes, f quasi-comp and quasi-separated. Then for any $\mathcal{F} \in QCoh(X)$
$f_* \mathcal{F} \in QCoh(Y)$

Pf. W.L.O.G Y is affine. Choose a finite affine covering $f^{-1}(Y) = \bigcup_{j=1}^{r} U_j$

As $X \to Y$ is quasi-separated, $U_i \cap U_j$ admits a finite affine open covering $U_i \cap U_j = \bigcup_{\alpha \in S_{ij}} V_\alpha$

Let $i_j : U_j \hookrightarrow X$, $i_\alpha' : V_\alpha \hookrightarrow X$ be the open immersions

We have an exact seq

$$0 \to \mathcal{F} \to \bigoplus (i_j)_* (\mathcal{F}|_{U_j}) \longrightarrow \bigoplus_{i,j} \bigoplus_{\alpha \in S_{ij}} (i_\alpha)_* \mathcal{F}|_{V_\alpha}$$

$$(s_j) \longmapsto (s_{j_1} - s_{j_2})|_{V_\alpha}$$

Applying f_* get an exact seq

$$0 \to f_* \mathcal{F} \to \bigoplus f_* (i_j)_* (\mathcal{F}|_{U_j}) \longrightarrow \bigoplus_{i,j} \bigoplus_{\alpha \in S_{ij}} f_* (i_\alpha)_* \mathcal{F}|_{V_\alpha}$$

As U_j, V_α, S are affine, $(f \circ i_j)_* \mathcal{F}|_{U_j}$ and $(f \circ i_\alpha)_* \mathcal{F}|_{V_\alpha}$ are q.coh $\forall\, \alpha, j$.

So $f_* \mathcal{F}$ is kernel of an $\mathcal{O}_Y$ linear map between q.coh sheaves.

So $f_* \mathcal{F}$ is q.coh.

12.11.2025 Lecture continued